Casimir force between some surfaces close to each other.

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Abstract

Casimir interactions (due to the scalar field fluctuations) of two surfaces which are close to each other are studied. After a brief general presentation, explicit calculations for co-axial cylinders, co-centric spheres and co-axial cones are performed.

I. Introduction

Experimentally the Casimir interactions have been observed for the geometrical setups involving two (actually disconected) surfaces [1]. The original Casimir force between parallel planes [2] was first verified by experiments in 1958 [3]. The experiments were repeated by higher precision for the force between the plates and the surfaces of different geometries i. e. cylinders and spheres [1]. Sphere above a plane configuration [4] is particularly convenient in experimental tests for it does not give rise the precise alignment problem of the parallel plate case [1]. Note that the calculation for the sphere-plane geometry is exact if the radius of the sphere is small compared to the distance to the plane [4]. Sphere-sphere geometry has also been studied, subject to the similar approximation as the sphere-plane problem [5]. The interactions of two concentric spheres has recently been addressed [6].

The purpose of the present work is to study new two surfaces geometries. We calculate the Casimir forces resulting from the vacuum fluctuations of

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scalar fields, between surfaces which are close to each other. Since the Casimir measurements are performed for small separ ations this approximation does not constitute a grave disadvantage.

After the brief outline of our approach to the surface-surface interaction in Section 2, we proceed with specific examples, that is co-axial cylinders, co-centric spheres and co-axial conical surfaces. In all three examples we get attractive Casimir forces.

II. Casimir energy for the region between two boundaries which are close to each other

We first choose the suitable spatial curvilinear coordinates η^j, $j = 1, 2, 3$ for the geometry we deal with. The corresponding Minkowski metric and the Klein-Gordon operator are then

$$ds^2 = dt^2 - g_{ij} d\eta^i d\eta^j \tag{1}$$

and ($g \equiv \det(g_{ij})$)

$$\triangle = \frac{\partial^2}{\partial t^2} - \frac{1}{\sqrt{g}} \frac{\partial}{\partial \eta^i} g^{ij} \sqrt{g} \frac{\partial}{\partial \eta^j}. \tag{2}$$

The Green function is

$$G = \sum_{\lambda_1, \lambda_2, \lambda_3} \frac{e^{i\omega(\lambda)(t-t')}}{2\omega(\lambda)} \Phi_{\omega(\lambda)}(\eta) \overline{\Phi_{\omega(\lambda)}(\eta')}, \tag{3}$$

where $\Phi_{\omega(\lambda)}(\eta)$ and $\omega^2(\lambda)$ are the eigenfunctions and eigenvalues of the equation for the massless scalar field

$$-\frac{1}{\sqrt{g}} \frac{\partial}{\partial \eta^i} g^{ij} \sqrt{g} \frac{\partial}{\partial \eta^j} \Phi_{\omega(\lambda)}(\eta) = \omega^2(\lambda) \Phi_{\omega(\lambda)}(\eta). \tag{4}$$

For massive scalar field one only changes ω^2 by $\omega^2 + \mu^2$; with μ being the mass. We assume that the above equation is separable in the spatial coordinates η^j. Here η and λ stand for the collection of the coordinates η^j and the corresponding eigennumbers λ_j (which are specified by the boundary conditions) respectively. The functions $\Phi_{\omega(\lambda)}(\eta)$ are normalized with respect to the norm

$$\|\Phi\|^2 = \int_A \sqrt{g} d^3\eta |\Phi(\eta)|^2, \tag{5}$$

where A is the domain of the coordinates η^j. For the renormalization purposes it is more convenient to study the Green function with the cut off factor $e^{\beta\omega(\lambda)}$:

$$G_\beta = \sum_{\lambda_1,\lambda_2,\lambda_3} e^{-\beta\omega(\lambda)} \frac{e^{i\omega(\lambda)(t-t')}}{2\omega(\lambda)} \Phi_{\omega(\lambda)}(\eta)\overline{\Phi_{\omega(\lambda)}(\eta')}, \tag{6}$$

The renormalized energy density is then obtained after calculating the coincidence limit derivatives as:

$$T = \ Ren \ [\lim_{t,\eta^j \to t', eta'^j} (\frac{\partial^2}{\partial t \partial t'} + g^{ij} \frac{\partial^2}{\partial \eta^i \partial \eta'^j}) G_\beta], \tag{7}$$

where (Ren) means taking the remaining finite part of the expression in the square bracket after $\beta \to 0$ limit.

To calculate the Casimir energy one needs the eigenvalues of the problem. The eigenvalues $\omega^2(\lambda)$ depend on tree quantum numbers λ_j corresponding to the degrees of freedoms in directions η^j in which we assume that the equation (4) can be separated. We further assume that after the separation of variables the eigenvalue equations in coordinates η^1, η^2 can be trivially solved, and the corresponding quantum numbers λ_1, λ_2 are easily obtained. This assumption does not introduce a strong restriction. In fact many problems in the literature are of that type. For example when one studies the Casimir energy inside a spherical cavity, only nontrivial problem is the radial problem in which one has to deal with the roots of the Bessel functions to impose the boundary condition [1].

In this work we employ an approximation method to calculate the nontrivial spectral parameter λ_3, which is valid if the problem involves two boundaries in direction η^3, which are close to each other.

After the separation, the problem in hand in η_3 should be of the following form

$$[-\frac{d}{d\eta^3} a(\eta^3) \frac{d}{d\eta^3} + V_{\lambda_1\lambda_2}(\eta_3)]\Phi_{\lambda_3}(\eta^3) = \omega^2(\lambda)\Phi_{\lambda_3}(\eta^3), \tag{8}$$

where $a(\eta^3)$ and $V_{\lambda_1\lambda_2}(\eta_3)$ are some functions determined by the explicit form of the metric (1). The effective potential $V_{\lambda_1\lambda_2}(\eta_3)$ in η^3 problem in general depends on the quantum numbers λ_1 and λ_2, which are the separation constants for variables η_1 and η_2. The boundary conditions we have to impose for the type of geometries under investigations are

$$\Phi_{\lambda_3}(\eta_0^3) = 0, \quad \Phi_{\lambda_3}(\eta_1^3) = 0, \tag{9}$$

where $\eta_0^3 < \eta_1^3$. In practice the above boundary conditions require dealing with the roots of special functions which are quite involved. However if the boundaries are closed to each other, i.e. for $\eta_1^3 - \eta_0^3 \ll \eta_0^3$, (as we explicitly see in the following sections) after bringing equation (8) into a Schrödinger form, one can approximate the effective potential with one which is easier to impose boundary conditions. In the examples we present, these approximated effective potentials are all constants.

III. Casimir energy in the region between the co-axial cylinders

In the cylindrical coordinates, i.e. with the metric

$$ds^2 = dt^2 - dz^2 - dr^2 - r^2 d\phi^2 \tag{10}$$

the eigenvalue problem we have to solve is

$$-[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}]\Phi = \omega^2 \Phi \tag{11}$$

After solving for the trivial coordinates z and ϕ we have

$$\Phi = \frac{e^{ipz+im\phi}}{2\pi} v_{nm}^p(r). \tag{12}$$

Here $v_{nm}^p(r)$ are the normalized wavefunctions corresponding to the radial equation

$$[-\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} + p^2 + \frac{n^2}{r^2}] v_{nm}^p = (\omega_{ln}^p)^2 v_{nm}^p. \tag{13}$$

They should satisfy the boundary conditions on the co-axial cylinders (with radii $r_0 < r_1$)

$$v_{nm}^p(r_0) = 0, \quad v_{nm}^p(r_1) = 0. \tag{14}$$

n is the radial quantum number corresponding to λ_3 of the previous section. The Green function with the cut off factor for the volume between the cylinders is given by

$$G_\beta = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} dp e^{-\beta\omega_{nm}^p}\frac{e^{i\omega_{nm}^p(t-t')+ip(z-z')+im(\phi-\phi')}}{8\pi^2\omega_{nm}^p}v_{nm}^p(r)v_{nm}^p(r'). \quad (15)$$

The renormalized vacuum energy density is then:

$$T = \ Ren \ [\lim_{t,r,z,\phi\to t',r',z',\phi'}(\partial_t\partial_{t'} + \partial_r\partial_{r'} + \partial_z\partial_{z'} + \frac{1}{r^2}\partial_\phi\partial_{\phi'})G_\beta]. \quad (16)$$

Substituting $v_{nm}^p = r^{-1/2}\psi_{nm}^p$ in (13)we obtain the Schrödinger equation (with $(\mu_{nm})^2 = (\omega_{nm}^p)^2 - p^2$)

$$[-\frac{d^2}{dr^2} + V(r)]\psi_{nm} = (\mu_{nm})^2\psi_{nm} \quad (17)$$

with the "potential"

$$V(r) = \begin{cases} \infty, & r = r_0, \ r = r_1 \\ \frac{m^2-\frac{1}{4}}{r^2}, & r \in (r_0, r_1) \end{cases} \quad (18)$$

Note that the solution of (17) satisfying the boundary condition at r_0 is given in terms of Bessel functions as

$$\psi_{nm}(r) = \frac{J_m(\mu_{nm}r_0)N_m(\mu_{nm}r) - J_m(\mu_{nm}r)N_m(\mu_{nm}r_0)}{\Omega_{nm}}, \quad (19)$$

where Ω_{nm} to be defined by the normalization

$$\int_{r_0}^{r_1} |\psi_{nm}(r)|^2 r dr = 1. \quad (20)$$

In practice however the above integral is very difficult to calculate for arbitrary values of $r_0, \ r_1$. The spectrum μ_{nm} should be determined from the boundary condition at r_1:

$$\psi_{nm}(r_1) = 0 \quad (21)$$

which is quite involved equation. However if the cylindrical surfaces are close to each other, we can relay on the approximation summarized in Section II.

If $\triangle r \equiv r_1 - r_0 \ll r_0$ we can approximate the effective potential in (18) by the constant one

$$V^0 = \{ \begin{array}{ll} \infty, & r = r_0,\ r = r_1 \\ \frac{m^2 - \frac{1}{4}}{R^2}, & r \in (r_0, r_1) \end{array} , \tag{22}$$

with R being some constant. The Schrödinger equations with the potential (22) is then trivially solved by

$$\psi^0_{nm} = \sqrt{\frac{2}{\triangle r}} \sin(\mu^0_{nm}(r - r_0)), \tag{23}$$

with eigenvalues

$$(\mu^0_{nm})^2 = \frac{\pi^2 n^2}{(\triangle r)^2} + \frac{m^2 - \frac{1}{4}}{R^2}; \quad n = 1, 2, 3, \ldots \tag{24}$$

The Condition for the approximation to be valid is

$$\| V(r) - V^0(r) \| \ll \min_n \{ (\mu^0_{nm})^2 \} = \frac{\pi^2}{(\triangle r)^2} + \frac{m^2 - \frac{1}{4}}{R^2}, \tag{25}$$

with

$$\| W \|^2 = \max_{(\psi,\psi)=1} (\psi, W W^\dagger \psi) \tag{26}$$

being the norm of operators in the Hilbert space of the Schrödinger equation 17). For the left hand side of the above inequality we use the estimation

$$\| V(r) - V^0(r) \| \leq |m^2 - \frac{1}{4}| max_{r \in [r_0, r_1]} |\frac{1}{r^2} - \frac{1}{R^2}| \tag{27}$$

For $R^2 = r_1 r_0$ the above conditions implie

$$\frac{\triangle r}{r_0} \ll 1. \tag{28}$$

We then insert the radial wave functions and the eigenvalues of (23) and (24) into (15), and then integrate the energy density (16) over the volume of the unit height between the cylinders. The total Casimir energy per unit height is then

$$E = \int_0^{2\pi} d\phi \int_{r_0}^{r_1} r dr T \tag{29}$$

or

$$E = -\ Ren\ [\frac{\partial}{\partial\beta}(\int_{-\infty}^{\infty}\frac{dp}{2\pi}\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} e^{-\beta\omega_{nm}^{p}})]. \tag{30}$$

After integration over p we have (with $\omega_{nm}^{p} = \sqrt{p^2 + (\mu_{nm})^2}$)

$$E = \frac{1}{\pi}\ Ren\ [\frac{\partial^2}{\partial\beta^2}(\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} K_0(\beta\mu_{nm})], \tag{31}$$

Here K_0 is the modified Bessel function. Employing the Plana formula [7]

$$\sum_{n=1}^{\infty} F(n) = \frac{F(0)}{2} + \int_0^{\infty} dt F(t) + i\int_0^{\infty} dt \frac{F(it) - F(-it)}{e^{2\pi t} - 1} \tag{32}$$

we have

$$\sum_{m=-\infty}^{\infty} K_0(\beta\mu_{nm}^{0}) = 2\int_0^{\infty} dt K_0(\frac{\beta}{R}\sqrt{t^2 + (\frac{\pi R n}{\triangle r})^2}) + 2\pi\int_{\frac{\pi R n}{\triangle r}}^{\infty} dt \frac{J_0(\frac{\beta}{R}\sqrt{t^2 - (\frac{\pi R n}{\triangle r})^2})}{e^{2\pi t} - 1}. \tag{33}$$

The first integral can explicitly be calculated [8]:

$$\int_0^{\infty} dt K_0(\frac{\beta}{R}\sqrt{t^2 + (\frac{\pi R n}{\triangle r})^2}) = \frac{\pi R}{2\beta} e^{-\beta\frac{\pi n}{\triangle r}}. \tag{34}$$

Since the range of t is larger or equal to $\frac{\pi R n}{\triangle r}$ in the second integral we write

$$\frac{1}{e^{2\pi t} - 1} \simeq e^{-2\pi t}. \tag{35}$$

We then obtain [8]:

$$\int_{\frac{\pi R n}{\triangle r}}^{\infty} dt \frac{J_0(\frac{\beta}{R}\sqrt{t^2 - (\frac{\pi R n}{x})^2})}{e^{2\pi t} - 1} = R\frac{e^{-\frac{\pi n}{\triangle r}\sqrt{(2\pi R)^2 + \beta^2}}}{\sqrt{(2\pi R)^2 + \beta^2}}. \tag{36}$$

We insert (34) and (36) into (33) then perform the n summation in (31). The energy between the cylinders per unit height is then

$$E = \frac{\pi^3 R}{\triangle r^3} A + \frac{1}{4\pi^3 R^2} D, \tag{37}$$

where

$$A = \;\; Ren \; [\frac{\partial^2}{\partial s^2} \frac{1}{s(e^s - 1)}]; \quad s \equiv \frac{\pi\beta}{\triangle r} \tag{38}$$

and

$$D = \;\; Ren \; [\frac{\partial^2}{\partial t^2} \frac{e^{-\frac{2\pi^2 R}{x}\sqrt{1+t^2}}}{\sqrt{1+t^2}}]; \quad t \equiv \frac{\beta}{2\pi R}. \tag{39}$$

To calculate A we write $e^s = 1 + s(1+\delta)$ with $\delta = \sum_{n=1}^{\infty} \frac{s^n}{(n+1)!}$. Then

$$A = - \;\; Ren \; [\frac{\partial^2}{\partial s^2} \frac{\delta + \delta^2 + \delta^3 + \delta^4}{s^2}] \tag{40}$$

The term δ^n with $n > 4$ give no contribution. It is also enough to take into account the terms up to order s^4 in δ. We then pick the finite terms and arrive at

$$A = -\frac{1}{2} - \frac{5}{36} - \frac{1}{8} - \frac{1}{60} \simeq -0,9. \tag{41}$$

Calculation for D gives

$$D = -\frac{2\pi^2 R}{\triangle r} e^{-\frac{2\pi^2 R}{\triangle r}} \tag{42}$$

which is negligible small for $\triangle r \ll R$.

The Casimir energy per unit height between the close cylinders is then

$$E = -0,9\pi^3 \frac{\sqrt{r_0 r_1}}{(\triangle r)^3} \tag{43}$$

which implies that the cylinders are attracted to each other by a radial force

$$F = -2,7\pi^3 \frac{\sqrt{r_0 r_1}}{(\triangle r)^4}. \tag{44}$$

IV. Casimir energy between two close concentric spheres

We employ the spherical coordinates

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{45}$$

and insert the solution in terms of the spherical harmonics

$$\Phi = Y^l_m(\theta,\phi)v^p_{nm}(r); \quad l = 0,1,2,\ldots, \ -l \le m \le l \tag{46}$$

into the Klein-Gordon equation (4). The resulting radial eigenvalue problem we have to deal is

$$[-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + \frac{(l+\frac{1}{2})^2}{r^2}]u_{ln} = (\omega_{ln})^2 u_{ln} \tag{47}$$

subject to the boundary conditions

$$u_{ln}(r_0) = 0, \quad u_{ln}(r_1) = 0. \tag{48}$$

Here $r_0 < r_1$ are the radii of the spheres and n is the radial quantum number to be determined by the boundary conditions. To satisfy the boundary conditions one has to deal with the roots of the radial wave function $u_{nl}(r)$ which as in the previous section are the Bessel functions (with m replaced by $l+1/2$). However since we are interested in $\triangle r \equiv r_1 - r_0 \ll r_0$ limit, we can proceed as we did in the previous section.

We first substitute $u_{ln} = r^{-1}\psi_{ln}$ in (47) and arrive at the Schrödinger equation (with boundary conditions (48)) which is same as the one in (17) with $m \to l+1/2$ replacement. The remaining steps are similar to the previous section. We obtain for the radial wave functions and the eigenvalues the following expressions:

$$\psi^0_{ln}(r) = \sqrt{\frac{2}{\triangle r}}\sin(\omega^0_{ln}(r-r_0)), \tag{49}$$

$$(\omega^0_{ln})^2 = \frac{\pi^2 n^2}{(\triangle r)^2} + \frac{(l+\frac{1}{2})^2}{R^2}; \quad n = 1,2,\ldots. \tag{50}$$

With the above approximated radial eigenfunctions and eigenvalues we can write the Green function with the cut off factor

$$G_\beta = \sum_{n=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} e^{-\beta\omega^0_{ln}}\frac{e^{i\omega_{ln}(t-t')}}{2\omega^0_{ln}}u^0_{ln}(r)u^0_{ln}(r')Y^l_m(\theta,\phi)\overline{Y^l_m(\theta',\phi')} \tag{51}$$

Integrating the renormalized vacuum energy density

$$T = \ Ren\ [\lim_{t,r,\theta,\phi\to t',r',\theta',\phi'}[\partial_t\partial_{t'} + \partial_r\partial_{r'} + \frac{1}{r^2}\partial_\theta\partial_{\theta'} + \frac{1}{r^2\sin^2\theta}\partial_\phi\partial_{\phi'}]G_\beta] \tag{52}$$

over the volume between two concentric spheres we get the total energy

$$E=\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \int_{r_0}^{r_1} r^2 drT. \tag{53}$$

The formula we get after the summation over $\sum_{m=-l}^{l}$ (using the addition theorem for the spherical harmonics) is

$$E=-\ Ren\ [\frac{\partial}{\partial\beta}(\sum_{n=1}^{\infty}\sum_{l=0}^{\infty}(2l+1)e^{-\beta\omega_{ln}^0})] \tag{54}$$

We then employ the Plana formula

$$\sum_{l=0}^{\infty}(l+\frac{1}{2})e^{-\beta\omega_{ln}^0}\simeq\int_0^\infty tdte^{-\frac{\beta}{R}\sqrt{t^2+(\frac{\pi Rn}{\triangle r})^2}}=\frac{R^2}{\beta}[\frac{1}{\beta}-\frac{\partial}{\partial\beta}]e^{-\beta\frac{\pi n}{\triangle r}} \tag{55}$$

to obtain

$$E=\frac{\pi^3R^2}{(\triangle r)^3}A, \tag{56}$$

where A is the same as (41). Thus the total energy is

$$E=-0,9\pi^3\frac{r_0r_1}{(\triangle r)^3} \tag{57}$$

which produces an attractive radial force

$$F=-2,7\pi^3\frac{r_0r_1}{(\triangle r)^4}. \tag{58}$$

We know that the Casimir energy in the sphere of radius R is positive [9]

$$E=\frac{0,9}{2R} \tag{59}$$

which implies a repulsive force

$$F_R=-\frac{\partial E}{\partial R}=\frac{0,9}{2R^2}. \tag{60}$$

The smaller the radius, the larger the force outward. Naively it is expected to have a net attraction between concentric spheres for outward sphere is less forced outward compared the inner one. However the force we obtain in (58) is very large compared to the difference of F_R and $R_{R-\triangle r}$, because in our case all contributions come from the very high frequency modes, for $\triangle r \ll R$.

V. Casimir energy between the co-axial cones

The eigenvalue problem in the spherical coordinates we have to solve is

$$-[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}]\Phi = \omega^2\Phi \tag{61}$$

subject to the boundary conditions (with $\theta_0 < \theta_1$)

$$\Phi|_{\theta=\theta_0} = 0, \quad \Phi|_{\theta=\theta_1} = 0 \tag{62}$$

where θ_0 and θ_1 are the azimuthal angles of two conic surfaces symmetric around the z- axis. Solution Φ is

$$\Phi^{\omega}_{nm} = \frac{e^{im\phi}}{\sqrt{2\pi}}\sqrt{\frac{p}{r}}J_{\lambda_{nm}+1/2}(\omega r)w_{nm}(\theta), \tag{63}$$

where w_{nm} is the solution of the eigenvalue equation

$$[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{m^2}{\sin^2\theta}]w_{nm}(\theta) = \lambda_{nm}(\nu_{nm}+1)w_{nm}(\theta) \tag{64}$$

subject to the boundary conditions

$$w_{nm}(\theta_0) = 0, \quad w_{nm}(\theta_1) = 0. \tag{65}$$

The solution of this boundary problem can be written in terms of the appropriate associated Legendre functions. The spectral parameter will be defined by the roots of this solution. The vacuum energy density is

$$T = \ Ren\ [\lim_{t,r,\theta,\phi\to t',r',\theta',\phi'}[\partial_t\partial_{t'} + \partial_r\partial_{r'} + \frac{1}{r^2}\partial_\theta\partial_{\theta'} + \frac{1}{r^2\sin^2\theta}\partial_\phi\partial_{\phi'}]G_\beta] \tag{66}$$

with

$$G_\beta = \sum_m\sum_n\int_0^\infty d\omega e^{-\beta\omega}\frac{e^{i\omega(t-t')}}{2\omega}\Phi^{\omega}_{nm}(\eta)\Phi^{\omega}_{nm}(\eta'). \tag{67}$$

Inserting the explicit form of wave functions (63) and using the formula [8]

$$\int_0^\infty d\omega e^{-\beta\omega} J_\nu(\omega r) J_\nu(\omega r) = \frac{1}{\pi r} Q_{\nu-1/2}(1+\frac{\beta^2}{2r^2}) \tag{68}$$

we obtain

$$E = \frac{1}{2\pi^2 r}\ Ren\ [\frac{\partial^2}{\partial\beta^2} + \frac{1}{4}\frac{\partial^2}{\partial r^2}] \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} \frac{Q_{\lambda_{nm}}(1+\frac{\beta^2}{2r^2})}{r} \tag{69}$$

where Q_λ is the Legendre function of the second kind. The difficulty is now finding the quantum number λ_{nm}. Note that unlike the previous examples studied in sections III and IV, the nontrivial quantum number is not ω which is integrated over but λ_{nm} which correspond to l of section IV. If the angle difference $\triangle\theta \equiv \theta_1 - \theta_0$ is very small the Schrödinger equation obtained from (64) after inserting $w_{nm} = \sin^{-1/2}\theta \psi_{nm}$

$$[-\frac{\partial^2}{\partial\theta^2} + \frac{m^2 - 1/4}{\sin^2\theta}]\psi_{nm}(\theta) = (\lambda_{nm} + 1/2)^2 \psi_{nm}(\theta) \tag{70}$$

can be approximated with the one with constant potential by the replacement

$$\frac{m^2 - 1/4}{\sin^2\theta} \to V_0(\theta). \tag{71}$$

The potential is

$$V_0(\theta) = \{ \begin{array}{ll} \infty, & \theta = \theta_0,\ \theta = \theta_1 \\ & \frac{m^2 - \frac{1}{4}}{\sin^2\Theta}, \end{array} \tag{72}$$

with Θ being some constant which will be specified later. Wave functions are again sinus functions and the spectrum is obtained as

$$(\lambda_{nm}^0 + 1/2)^2 = (\frac{\pi n}{\triangle\theta})^2 + \frac{m^2 - 1/4}{\sin^2\Theta}, \quad n = 1, 2, \ldots \tag{73}$$

The condition for the approximation to be valid is

$$\| \frac{m^2 - \frac{1}{4}}{\sin^2\Theta} - \frac{m^2 - \frac{1}{4}}{\sin^2\theta} \| \ll (\lambda_{1m}^0 + 1/2)^2, \tag{74}$$

which for $\Theta = \theta_0$ implies

$$\frac{\sin(2\theta_0)\triangle\theta}{\sin^2\theta_1} \ll 1. \tag{75}$$

Due to this restriction we have also

$$\lambda^0_{nm} \simeq \sqrt{(\frac{\pi n}{\triangle\theta})^2 + \frac{m^2}{\sin^2\Theta}}. \tag{76}$$

We employ the above λ^0_{nm} in (69). Summation over m and n is

$$\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} Q_{\lambda^0_{nm}}(z) \simeq 4Rx\int_0^\infty sdsQ_s(z) - R\int_0^\infty dsQ_s(z) \tag{77}$$

with $z = 1 + \frac{\beta^2}{2r^2}$ Using the expression

$$\sum_{m=0}^{\infty} t^m Q_m(z) = \frac{1}{z-t}\log\frac{2(z-t)}{\sqrt{z^2-1}}), \tag{78}$$

with $z > 1$ and $t < 1$ we have

$$\int_0^\infty dsQ_s(z) \simeq \frac{1}{z-1}\log(2\sqrt{\frac{z-1}{z+1}}) \tag{79}$$

and

$$\int_0^\infty sdsQ_s(z) \simeq \frac{1}{(z-1)^2}(\log(2\sqrt{\frac{z-1}{z+1}}) - 1) \tag{80}$$

which are valid for z being very close to 1. After some elaboration we get the vacuum energy density between the close conical surfaces:

$$E = \frac{3\triangle\theta}{32\pi^2 r^4}. \tag{81}$$

The total energy in the slice of unit thickness in z-axis, at angle θ is

$$E = 2\pi r\sin\theta\frac{3\triangle\theta}{32\pi^2 r^4}. \tag{82}$$

This energy causes an attractive torque on the circular belt of unit thickness along the z-axis

$$T = -\frac{\partial E}{\partial\triangle\theta} = -\frac{3\sin\theta}{16\pi r^3}. \tag{83}$$

It is interesting to compare the attractive force we obtained with the repulsive Casimir force between two planes with the angle $\alpha < \pi$ between them, i.e.

the wedge problem [9]. The torque at the distance r from the intersection line of the planes on the unit area is

$$F_\alpha = -\frac{\partial}{\partial\alpha}\frac{1}{144r^4\alpha^2}(\frac{\pi^2}{\alpha^2} - \frac{\alpha^2}{\pi^2}) = \frac{\pi^2}{380r^4\alpha^5} \tag{84}$$

is repulsive. However if we consider four planes with a common intersection line with the angle α between the inner planes and with the angle β between the outer and the inner ones, the net torque on the outer two planes will be:

$$F_\alpha - 2F_\beta = \frac{\pi^2}{380r^4}(\frac{1}{\alpha^5} - \frac{2}{\beta^5}) \tag{85}$$

which is attractive for $\beta \ll \alpha$. This is in qualitative agreement with our example. However we should not forget the significant topological difference of our geometry.

VI. Final Remarks

The geometries we studied which are surfaces close to each other are all locally same as the parallel plane geometry. However, although the forces we obtained are all attractive, quantitatively they are quite different from the forces between the parallel planes. This differences we think are due to the global differences of the geometries.

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